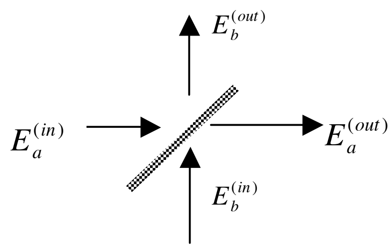
Physics 566, Quantum Optics
Problem Set #6
Due: Monday Nov. 8, 2010

Problem1: The beam splitter and other linear transformations (20 points)
We're all familiar with classical linear optics. This problem explores the quantum description.

Consider a symmetric beam splitter



The pair $\left(E_a^{(out)}, E_b^{(out)}\right)$ is related to $\left(E_a^{(in)}, E_b^{(in)}\right)$ through a unitary "scattering matrix"

$$\begin{bmatrix} E_a^{(out)} \\ E_b^{(out)} \end{bmatrix} = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} E_a^{(in)} \\ E_b^{(in)} \end{bmatrix}$$

(a) **Show** that $|t|^2 + |r|^2 = 1$, $Arg(t) = Arg(r) \pm \dfrac{\pi}{2}$, so that a possible transformation is,

$$E_a^{(out)} = \sqrt{T}\, E_a^{(in)} + i\sqrt{1-T}\, E_b^{(in)}, \quad E_b^{(out)} = \sqrt{T}\, E_b^{(in)} + i\sqrt{1-T}\, E_a^{(in)}, \text{ where } T = |t|^2.$$

Classically, if we inject a field only into one input port, leaving the other empty, the field in that mode will become attenuated, e.g., $E_a^{(out)} = \sqrt{T}\, E_a^{(in)} < E_a^{(in)}$.

(b) Consider now the quantized theory for these two modes, $E_a \Rightarrow \hat{a}$, $E_b \Rightarrow \hat{b}$. Suppose again that a field is injected only into the "a-port". Show that

$\hat{a}^{(out)} = \sqrt{T}\hat{a}^{(in)}$ is inconsistent with the quantum uncertainty.

(c) In order to preserve the proper commutation relations we cannot ignore *vacuum fluctuations* entering the unused port. **Show that** if the "in" and "out" creation operators are related by the scattering matrix,

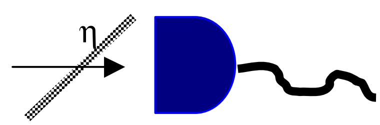
$$\begin{bmatrix} \hat{a}^{(out)\dagger} \\ \hat{b}^{(out)\dagger} \end{bmatrix} = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} \hat{a}^{(in)\dagger} \\ \hat{b}^{(in)\dagger} \end{bmatrix},\ \text{the commutator is preserved.}$$

(d) Suppose a single photon is injected into the a-port, so that the "in-state" is $\left|\psi^{(in)}\right\rangle = |1\rangle_a \otimes |0\rangle_b$. The "out-state" is $\left|\psi^{(out)}\right\rangle = \hat{S}\left|\psi^{(in)}\right\rangle$ where $\hat{S}$ is the "scattering operator", defined so that $\hat{S}\hat{a}^{(in)\dagger}\hat{S}^\dagger = \hat{a}^{(out)\dagger}$ and $\hat{S}\hat{b}^{(in)\dagger}\hat{S}^\dagger = \hat{b}^{(out)\dagger}$.

Show that $\left|\psi^{(out)}\right\rangle = t|1\rangle_a \otimes |0\rangle_b + r|0\rangle_a \otimes |1\rangle_b$.

(e) Suppose a coherent state is injected into the a-port $\left|\psi^{(in)}\right\rangle = |\alpha\rangle_a \otimes |0\rangle_b$. Which is the output, $\left|\psi^{(out)}\right\rangle = |t\alpha\rangle_a \otimes |r\alpha\rangle_b$ or $\left|\psi^{(out)}\right\rangle = r|\alpha\rangle_a \otimes |0\rangle_b + t|0\rangle_a \otimes |\alpha\rangle_b$? Explain the difference between these.

(f) We can model a photon counter with a finite quantum efficiency η as perfect detector preceded by a beam splitter of with transmission coefficient η.



Show that the photon counting statistics, i.e. the probability to detect *m* photons is
$P_m(\eta) = \sum_{n=m}^{\infty} p_n \binom{n}{m} \eta^m (1-\eta)^{n-m}$, where p_n is the distribution before the beam-splitter.

Explain the meaning of this expression.

(g) A general linear optical system consisting, e.g., of beam-splitters, phase shifters, mirrors, etalons, etc. can be described by a unitary transformation on the modes

$$E_k^{(out)} = \sum_{k'} u_{kk'} E_{k'}^{(in)} .$$

In the quantum description the mode operators transform by the scattering transformation
$\hat{a}_k^{(out)} = \hat{S}\hat{a}_k^{(in)}\hat{S}^\dagger = \sum_{k'} u_{kk'}\hat{a}_{k'}^{(in)}$, where $u_{kk'}$ is a unitary matrix.

Show that if we start with a multimode coherent state $\left|\psi^{(in)}\right\rangle = \left|\left\{\alpha_k^{(in)}\right\}\right\rangle$, the output state is ALSO a coherent state, $\left|\psi^{(out)}\right\rangle = \left|\left\{\alpha_k^{(out)}\right\}\right\rangle$, with $\alpha_k^{(out)} = \sum_{k'} u_{kk'}\alpha_{k'}^{(in)}$.

(i) The previous part highlights how linear transformations are essentially classical. This was true for exactly one photon inputs or coherent states. However, this is not true for more general inputs. Suppose we send one photon into *both ports*, of a 50-50 beam-splitter T=1/2, $\left|\psi^{(in)}\right\rangle = \left|1\right\rangle_a \otimes \left|1\right\rangle_b$. **Show that** the output state is,

$$\left|\psi^{(out)}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|2\right\rangle_a\left|0\right\rangle_b + \left|0\right\rangle_a\left|2\right\rangle_b\right).$$

This says that the two photons "bunch", both going to port-a or to port-b, but never one in port-a and one in port-b. This is an effect of Bose-Einstein quantum statistics. **Explain** in terms of destructive interference between indistinguishable processes.

Problem 2: Resonance fluorescence from a two-level atom (20 points)

Consider a two-level atom driven by a classical electric field, near resonance. The atom will scatter this radiation, i.e., it will fluoresce. The purpose of this problem is to study some of the import quantum optical properties of this radiation.

We measure the scattered electric field with a detector at some position $\mathbf{r}_D$ and the atom at the origin. As seen in Problem 4 of P.S.#4, the radiated field operator is proportion to the atomic dipole operator at the retarded time

$$E^{(\pm)}(\mathbf{r}_D,t) \propto d^{(\pm)}(t - r_D / c) \propto \sigma_{\mp}(t - r_D / c)$$

where $\sigma_{\mp}(t)$ is the atomic lowering/raising operator (in the Heisenberg picture).

(a) Show that the mean intensity measured by the detector proportional to

$$\langle I(t)\rangle \propto \left\langle E^{(-)}(\mathbf{r}_D,t)E^{(+)}(\mathbf{r}_D,t)\right\rangle \propto \left\langle \sigma_-(t - r_D / c)\sigma_+(t - r_D / c)\right\rangle = P_e(t - r_D / c)$$

where $P_e(t)$ is the probability of being in the excited state at time *t*. *Interpret this result.*

In steady state, when detailed balance is achieved between absorption and emission after a few spontaneous decay lifetimes, $P_e(t)$ goes to constant (stationary statistics).
What is $P_e(t)$ in steady state from our solution to the optical Bloch equations?

(b) The radiating dipole has a mean value and fluctuations about the steady state solution due to the vacuum. Let us write, $\sigma_{\pm}(t - r_D / c) = \langle \sigma_{\pm}(t - r_D / c)\rangle + \delta\sigma_{\pm}(t - r_D / c)$, where the expectation value is taken in the steady state solution and the fluctuation operator is then given by $\delta\sigma_{\pm}(t - r_D / c) = \sigma_{\pm}(t - r_D / c) - \langle \sigma_{\pm}(t - r_D / c)\rangle$.

- Show that in steady state, $\langle I \rangle \propto \left|\langle \sigma_+ \rangle\right|^2 + \langle \delta\sigma_+ \delta\sigma_- \rangle$.
- The term $\left|\langle \sigma_+ \rangle\right|^2 \equiv I_{coh}$ is defined as the "coherent" (or elastic) part of the scattered intensity. *Justify that interpretation.*

  Show that $I_{coh} = \dfrac{1}{2}\dfrac{s}{(1+s)^2}$, where *s* is the saturation parameter.

(c) The part arising from the fluctuating dipole, $\langle \delta\sigma_+ \delta\sigma_- \rangle \equiv I_{incoh}$, is known as incoherent (or inelastic) component of the scattered intensity.

- *Show that* $I_{incoh} = \dfrac{1}{2}\dfrac{s^2}{(1+s)^2}$. Note that the ratio $I_{incoh} / I_{coh} = s$, thus for low saturation, elastic coherent photon scattering dominates over incoherent inelastic scattering.
- Plot I_{coh} and I_{incoh} as a function of *s*.

Aside: The two-time, first-order autocorrelation of the field determines the *power spectrum* of scattered light, as discussed in lecture. Fourier transforming the correlation function, the spectrum shows an "elastic peak" at the frequency of the incident light, with power given by I_{coh}, and the famous "Mollow triplet", consisting of three peaks at frequencies $\omega = \omega_0, \omega_0 \pm \Omega$, where ω_0 is the atomic resonance, and Ω is the Rabi frequency. This part of the spectrum represents the "inelastic scattering". This is discussed in all textbooks on quantum optics.

Consider now the second-order correlation function, representing the correlation between detecting photons at two different times,

$$G^{(2)}(t+\tau,t) = \left\langle E^{(-)}(t)E^{(-)}(t+\tau)E^{(+)}(t+\tau)E^{(+)}(t)\right\rangle .$$

After a short transient period, when the atom reaches steady state,

$$G^{(2)}(\tau) \propto \left\langle \sigma_+(0)\sigma_+(\tau)\sigma_-(\tau)\sigma_-(0)\right\rangle = Tr\left(\rho_0\sigma_+(0)\sigma_+(\tau)\sigma_-(\tau)\sigma_-(0)\right),$$

where ρ_0 is the steady-state solution (equivalent to the "initial time" here).

(d) *Show that* $G^{(2)}(\tau) \propto \left\langle \pi_e(\tau)\pi_g(0)\right\rangle P_{e,0}$, where $P_{e,0}$ is the probability to be in the excited state at the start of the correlation and, $\pi_g(0) = |g\rangle\langle g|$, $\pi_e(0) = |e\rangle\langle e|$.

(Note: Because the time evolution of the atoms alone is not unitary, this is only true in the Markov approximation discussed in the Wigner-Weisskopf approximation. This expression for $G^{(2)}$ follows from what is known as the "quantum regression theorem)

(e) It follows from (d) that, $G^{(2)}(\tau) \propto P(e;\tau \mid g;0)P_{e,0}$, where $P(e;\tau \mid g;0)$ is the conditional probability that the atom is in the excited state at time τ, given that it was in the ground state at time 0. *Explain the physical meaning of this result.*

(f) The expression for $P(e;\tau \mid g;0)$ follows from the solution to the optical Bloch equations for damped Rabi oscillations (known as the Torrey solution). We did not do this in class, but it follows straightforwardly from the Bloch eqns. On resonance,

$$\left\langle \sigma_z(t)\right\rangle = -1 + \frac{\Omega^2}{\Omega^2 + \Gamma^2/2}\left[1 - e^{-(3\Gamma/4)t}\left(\cos\tilde{\Omega}t + \frac{3\Gamma}{4\tilde{\Omega}}\sin\tilde{\Omega}t\right)\right],$$

where Γ is the spontaneous emission rate and $\tilde{\Omega} \equiv \sqrt{\Omega^2 - \Gamma^2/4}$.

Use this to show that the normalized correlation function is

$$g^{(2)}(\tau) = 1 - e^{-(3\Gamma/4)\tau}\left(\cos\tilde{\Omega}\tau + \frac{3\Gamma}{4\tilde{\Omega}}\sin\tilde{\Omega}\tau\right)$$

(h) Plot this as a function of $\Gamma\tau$ for the cases $2\Omega/\Gamma = .1$ and $2\Omega/\Gamma = 10$. Comment on the important features of these curves.